Quantitative Macroeconomics I
TD 3: Value Function Iteration

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Today's Agenda

1. Feedback & questions about Problem Set I

2. **Theory**: Bellman Equation, example of a consumption-saving program (reminder)
   - Recursive form of the deterministic problem
   - Markov chains and stochastic dynamic programming
   - Contraction mapping theorem and backward iteration

3. **Computational**: Value Function Iteration

   Global method, on the state space, can study uncertainty
   - On-grid Value Function Iteration
   - Off-grid VFI & Euler errors

   $\Rightarrow$ Pseudo-code of the algorithm (whiteboard)

A consumption saving program, without uncertainty

For the PS, you will be asked to solve a Real Business Cycles model. In this tutorial, we will take the example of a consumption (c_t) saving (a_{t+1}) program in partial eq. You will have to think carefully about the differences between the two models for the problem set…

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$$V_t(a_t) \equiv \max_{\{c_s, a_{s+1}\}_{\forall s \geq t}} \sum_{s \geq t} \beta^{s-t} u(c_s) \quad \text{s.t.} \quad \begin{cases} c_s + a_{s+1} = (1 + r_s) a_s + \bar{y} & \forall s \geq t \\ a_s \geq 0 & \forall s \geq t \\ a_t \quad \text{is given} & \end{cases}$$

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Issue: This problem is subject to the curse of dimensionality...

$\rightarrow$ How would you reduce its dimensionality to find a global solution?

Dimensionality reduction of the problem

1/ Algebra: reduce the number of control variables using the budget constraint

$$V_t(a_t) = \max_{\{a_{s+1}\}_{\forall s \geq t}} \sum_{s \geq t} \beta^{s-t} u\left((1+r_s)a_s + \bar{y} - a_{s+1}\right) \quad \text{s.t.} \quad a_{s+1} \geq 0 \quad \forall s \geq t$$

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2/ We can write the problem in the **state space** (recursive form)

$$V_t(a_t) = \max_{a_{t+1}} \left\{ u(c_t) + \beta \underbrace{\left[\max_{\{a_{s+1}\}_{\forall s \geq t+1}} u(c_{t+1}) + \sum_{s \geq t+1} \beta^{s-(t+1)} u(c_s) \right]}_{V_{t+1}(a_{t+1})} \right\}$$

$$\text{s.t.} \quad c_t = (1+r_t)a_t + \bar{y} - a_{t+1} \qquad \text{and} \quad a_{t+1} \geq 0$$

The deterministic Bellman Equation

Bellman Equation reduces the problem to ”today’s choice” given ”tomorrow” optimal

$$V_t(a_t) = \underbrace{\mathcal{T}\{V_{t+1}\}(a_t)}_{\text{Bellman operator}} = \max_{a_{t+1}} u\left((1+r)a_t + \bar{y} - a_{t+1}\right) + \beta V_{t+1}(a_{t+1}) \qquad \text{s.t.} \quad a_{t+1} \geq 0$$

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3. $a_{t+1} \in \Gamma(a_t)$ is the **Choice correspondence**
4. The Bellman Equation maps a function into a function. It is a ***functional equation***

Deriving the Euler Equation using the Bellman Equation

$$V_t(a_t) = \max_{c_t, a_{t+1}} \left\{ u(c_t) + \beta V_{t+1}(a_{t+1}) \right\} \quad \& \quad a_{t+1} = \bar{y} + R \times a_t - c_t$$

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1. Taking the FOC on c_t yields

$$u'(c_t) + \beta \times \frac{\partial V_{t+1}}{\partial a_{t+1}} \underbrace{\frac{\partial a_{t+1}}{\partial c_t}}_{=-1} = 0 \iff u'(c_t) = \beta \times \frac{\partial V_{t+1}}{\partial a_{t+1}}$$

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2. Use the Envelope theorem

   More details: `https://www.econ2.jhu.edu/people/ccarroll/public/lecturenotes/consumption/Envelope/`

$$\frac{\partial V_t}{\partial a_t} = \frac{\partial}{\partial a_t} \Big[u((1+r)a_t + \bar{y} - a_{t+1}) + \beta V_{t+1}(a_{t+1}) \Big] \Big|_{a_{t+1} = a^*_{t+1}(a_t)} = \frac{\partial u(c_t)}{\partial c_t} \times \frac{\partial c_t}{\partial a_t} = R \times u'(c_t)$$

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3. Combining both yields the **Euler Equation**:

$$u'(c_t) = \beta(1+r)u'(c_{t+1})$$

Adding risk: Markov chains

Risk is a key feature of economic behavior, often modeled using **Markov processes**

$\hookrightarrow$ In our example, take households' earning, subject to risk $\omega \sim AR(1)$

Formally, uncertainty is different from risk. It is often modelled as a risk on the variance of the AR1 process. Ask Moritz to know more!

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2/ Markov chains are defined by

- A discrete set of states Ω, with a probability transition matrix $\Pi = (\pi_{\omega,\omega'})_{\forall \omega,\omega' \in \Omega^2}$
- An initial distribution μ_0

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Stochastic Bellman Equation

Earnings are stochastic $\rightarrow$ replace $\bar{y}$ by ω. The Bellman Equation becomes

$$
\begin{aligned}
V_t(\omega_t, a_t) &= \mathcal{T}\{V_{t+1}\}(\omega_t, a_t) \\
&= \max_{a_{t+1}} u\left((1+r)a_t + \omega_t - a_{t+1}\right) + \beta \mathbb{E}_{\omega_{t+1}|\omega_t} V_{t+1}(\omega_{t+1}, a_{t+1}) \\
&= \max_{a_{t+1}} u\left((1+r)a_t + \omega_t - a_{t+1}\right) + \beta \sum_{\omega_{t+1}} \pi_{\omega_t,\omega_{t+1}} V_{t+1}(\omega_{t+1}, a_{t+1}) \\
\text{s.t.} \quad & a_{t+1} \geq 0 \qquad \forall\ \omega_t, a_t \in \Omega \times \mathbb{R}_+
\end{aligned}
$$

Two useful applications of the Bellman Equation

1. **At steady state**, the Value Function is the unique[1] fixed point to the Bellman operator

$$\text{CMT} : V^* \text{ s.t. } V^*(\omega, a) = \mathcal{T}\{V^*\}(\omega, a) = \max_{a'} u\left((1+r)a + \omega - a'\right) + \beta \mathbb{E}_{\omega'|\omega} V^*(\omega', a')$$

[1] By the application of the contraction mapping theorem. See the lecture slides.

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2. **Backward induction** (in finite time):

   1/ Start from a terminal condition $V_{T+1}(a_{T+1})$ e.g $V_{T+1}(a_{T+1}) = 0$ if T is large enough, HH is dead after T

   2/ Given a sequence of ω_t, the Value function at time $t < T+1$ is obtained by applying the Bellman Operator

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$$V_t(\omega_t, a_t) = \mathcal{T}\{V_{t+1}\}(\omega_t, a_t) \qquad \forall a_t$$

$\Rightarrow$ We can find the sequence of optimal policy functions $\{a'_t(a_t)\}_{t=0}^{T}$

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Stationary Value Function Iteration (VFI)

How to find the value function and policy functions at steady state?

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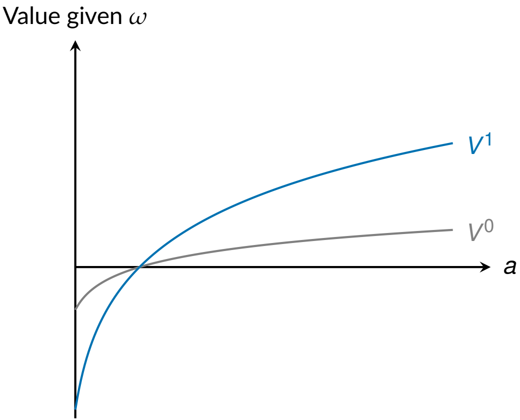

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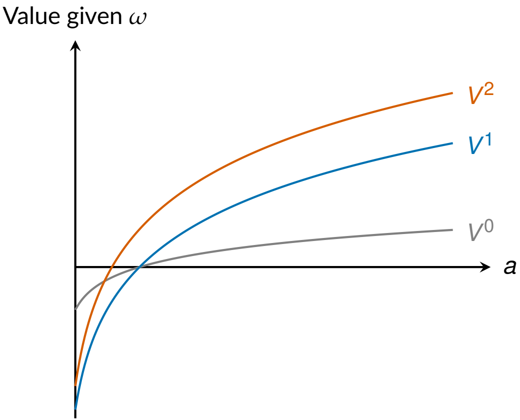

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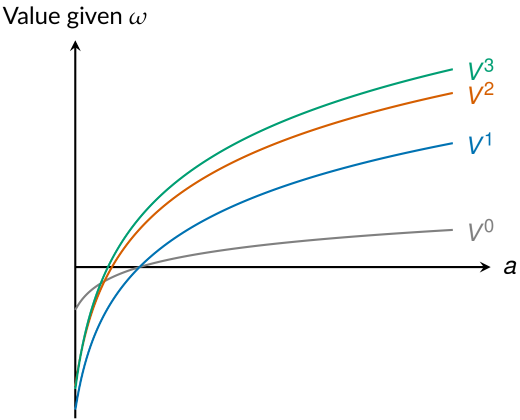

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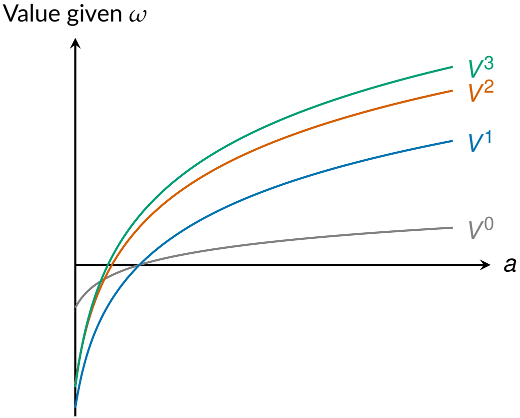

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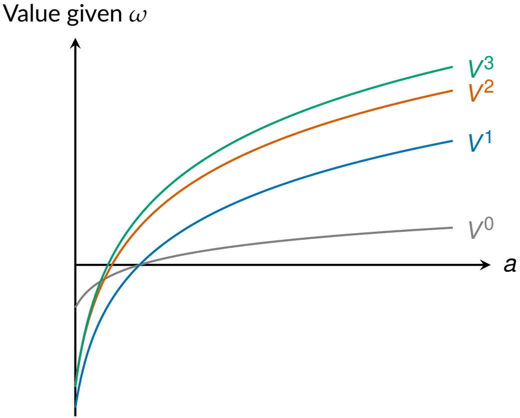

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$\Longrightarrow$ Let's do a pseudo-code of the algorithm together!

Pseudo-code for stationary VFI – Roadmap

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Goal: Find the fixed point of the Bellman equation $V^*(\omega, a) = \mathcal{T}\{V^*\}(\omega, a)$

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4. We need to iterate until a convergence criterion/measure is met $\rightarrow$ kind of norm?
5. We need to store results (VF, PF), and check if they make sense $\rightarrow$ Plotting, Euler errors?

Storing Value Functions and Policy Functions in MATLAB

VFI consists in iterating on the value function by applying the Bellman operator

- At each iteration, we need to **store the value function**

  Remember, V : state space $\rightarrow \mathbb{R}$, but the state space is continuous!

  $\Rightarrow$ Solution: **discretize** the state space into **grids** of points $(\omega_i, a_j) \in \mathcal{G}_\omega \times \mathcal{G}_a$

Note: You could also use projection methods to store these functions (e.g. Chebyshev polynomials), see Tobias' slides

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- The value function is stored as a **matrix** $\mathbf{V} = (V(\omega_i, a_j))_{\forall i,j}$ (size: $N_\omega \times N_a$)

  The same holds for the policy functions!

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- The value function is stored as a **matrix V** $= (V(\omega_i, a_j))_{\forall i,j}$ (size: $N_\omega \times N_a$)

  The same holds for the policy functions!

$\Rightarrow$ Convergence between two matrices (*discretized value functions*) using a norm

$$||\cdot||_\infty = \max_{i,j} |V^{(2)}(\omega_i, a_j) - V^{(1)}(\omega_i, a_j)| \qquad \text{with} \quad \omega_i, a_j \in \mathcal{G}_\omega \times \mathcal{G}_a$$

Note: You could also use projection methods to store these functions (e.g. Chebyshev polynomials), see Tobias' slides

Coding the Bellman operator $\mathcal{T}$

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3/ How to solve this problem numerically? Two options:

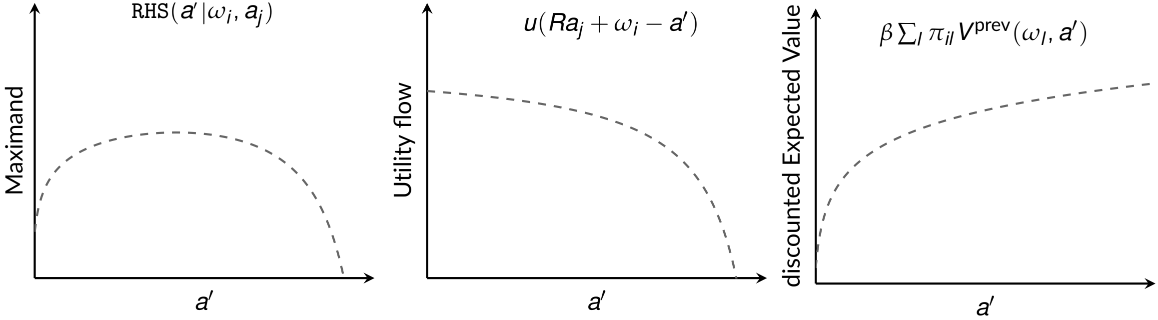
(a) On-grid VFI: restrict a' to be on the grid $\mathcal{G}_a$ (grid search) $\rightarrow$ **extremely imprecise**

(b) Off-grid VFI: use a numerical optimizer (e.g. golden-section search), **interpolate** between grid points

Diving into the RHS of the Bellman – given $\omega = \omega_i$ & $a = a_j$

Maximand of the RHS of the Bellman Equation

$$\text{RHS}(a'|\omega_i, a_j) = u(Ra_j + \omega_i - a') + \beta \sum_{\omega'} \pi_{\omega_i,\omega'} V^{\text{prev}}(\omega', a')$$



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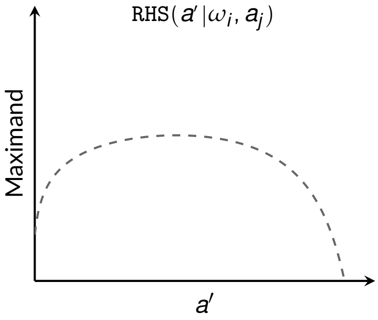

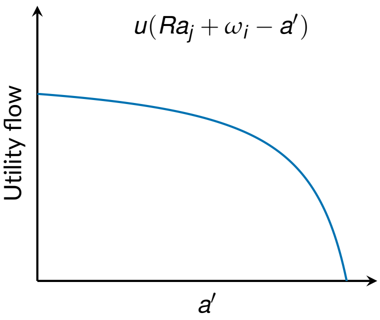

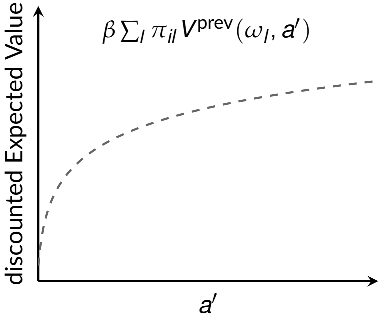

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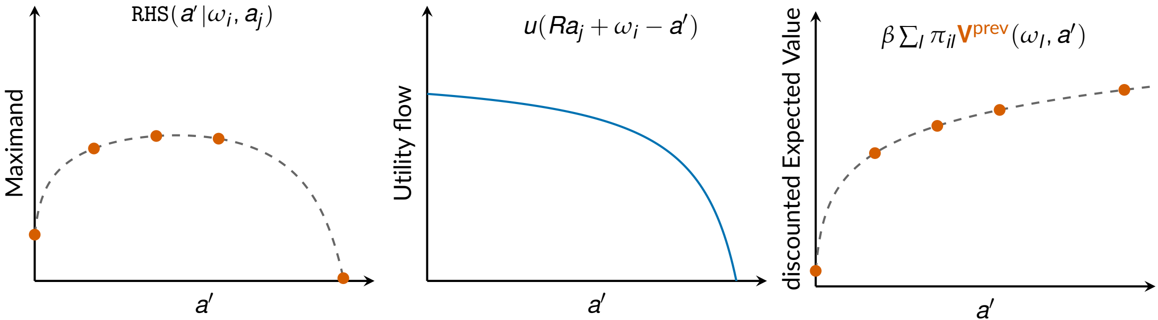

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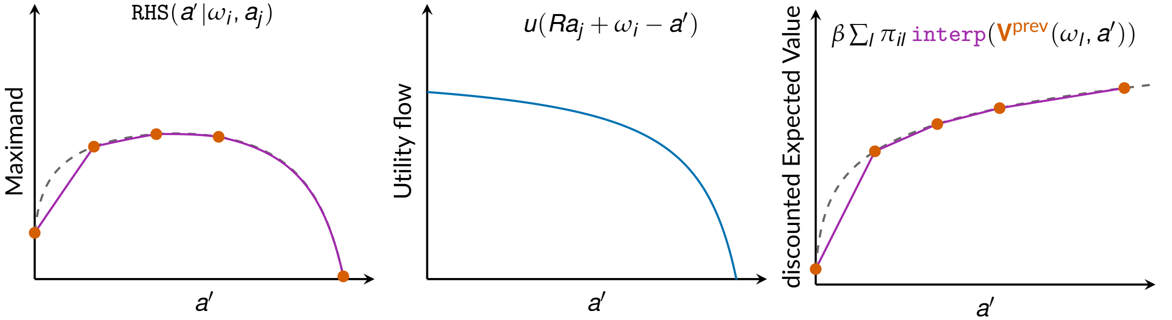

Interpolation

Linear interpolation of $V^{\text{prev}}(\omega')$ at a'

1. Find m such that $a_m \leq a' \leq a_{m+1}$ (extrapolation: if $a' \geq a_{N_a}$ take $m = N_a - 1$)
2. Compute $t = \frac{a' - a_m}{a_{m+1} - a_m}$
3. Interpolate: $V^{\text{prev}}(\omega', a') \approx (1 - t) \cdot V^{\text{prev}}(\omega', a_m) + t \cdot V^{\text{prev}}(\omega', a_{m+1})$

- Linear interpolation preserves monotonicity and concavity/convexity.
- Can approximate non-linear functions if the grid is dense enough where the function has high curvature
- Interpolator is piecewise linear, continuous, but not differentiable at grid points

Note: You may use later other interpolation methods (e.g. spline). Have a look at Fatih Guvenen's lecture 2 for more details.

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   Baseline: inside the loop on $\mathcal{G}_\omega \times \mathcal{G}_a$, at each evaluation of maximand, compute $\sum_j \pi_{i,l} V^{\text{prev}}(\omega_l, a')$

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   Advantage: the maximand uses a single linear interpolation, cheap to evaluate!

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      Advantage: pre-compute only once per iteration on VF! Be very careful with dimensions if you have an additional state

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   (c) Parallelize the outer loop (over $\mathcal{G}_\omega$) using `parfor`. Not always faster! More in Jesus Fernandez Villaverde's slides.

Accuracy: Tolerance and precision levels

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  $\Rightarrow$ Typical values: $\varepsilon = 10^{-4}$ (low precision), 10^{-6} (medium), 10^{-8} (high)

  Note: The smaller ε, the longer the computation time (more iterations). But a small ε is not sufficient to guarantee a precise solution!

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Checking Precision: Euler Errors

After convergence, compute the **Euler errors** at each point of the state space, using the policy functions

$$\mathrm{EE}(\omega_i, a_j) = \left| 1 - \frac{u'(c(\omega_i, a_j))}{\beta(1+r)\sum_l \pi_{i,l} \times u'\left(c\left(\omega_l, a'(\omega_i, a_j)\right)\right)} \right|$$

Additional references for this TD session

Main references:

- Heer & Maussner (2022), *DGE modelling*, 2nd edition, Chapters 4.1 and 4.2
- Azzimonti *et al.* (2025), *Macroeconomics*, Chapters 4.4 and 10.3, 10.4, 10.5 (link here)

Other references:

- Xin Yi's lecture notes on dynamic programming (link here)

  Nice starting point if you are lost
- Feodor Ishakov's lecture 40 on VFI (link here)

  Even has a youtube video explaining the process!
- QuantEcon's notebook on the stochastic growth model (link here)
- *[Advanced]* Fatih Guvenen's slides on dynamic programming and VFI (lectures 1, 2, 5: link here)