Quantitative Macroeconomics I
Bootcamp 2: Shooting Method

Grégoire Sempé

gregoire.sempe@psemail.eu

Paris School of Economics, Université Paris 1 Panthéon-Sorbonne

September 22, 2025

I thank Tobias Broer, Eustache Elina and Moritz Scheidenberger for useful materials and discussions.

Today's Agenda

1/ Framework: Neoclassical Growth Model

- Model presentation & main equations
- System of first difference equations & phase diagram

2/ How to solve the model using a shooting algorithm?

$\Rightarrow$ First problem set (ungraded) on shooting method for next week!

The Ramsey Growth Model

- The Ramsey model is the basis of all modern macroeconomic models, and is simple!
  - Representative agent model with endogenous saving rate
  - Perfect competition & no friction: decentralized solution = social planner solution
  - Consider a discrete time version $t \in \mathbb{N}_+$
  - Parameters: $\alpha \in (0,1)$, $\beta \in (0,1)$, $\delta \in (0,1)$, $\sigma \in \mathbb{R}_+ \backslash \{1\}$

- Perfectly competitive firm produces a generic good from capital $f(k) = k^\alpha$

- Representative household maximizes its flows of utility over time: $\sum_{t=0}^{\infty} \beta^t u(c_t)$

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

- Resource constraint: $c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$

- Two versions: finite horizon $T < \infty$ or infinite horizon $T = \infty$

Check the main slides of the course for more details...

The sequence problem

$$V_0(k_0) = \max_{\{c_t, k_{t+1}\}_{\forall t \geq 0}} \sum_{t \geq 0} \beta^t u(c_t) \quad \text{s.t.} \begin{cases} c_t + k_{t+1} = f(k_t) + (1-\delta)k_t & \forall t \geq 0 \\ k_0 \text{ given} & \\ c_t \geq 0, k_{t+1} \geq 0 & \forall t \geq 0 \end{cases}$$

Need to add the transversality condition to define the problem properly...

The sequence problem

$$V_0(k_0) = \max_{\{c_t, k_{t+1}\}_{\forall t \geq 0}} \sum_{t \geq 0} \beta^t u(c_t) \quad \text{s.t.} \begin{cases} c_t + k_{t+1} = f(k_t) + (1-\delta)k_t & \forall t \geq 0 \\ k_0 \text{ given} & \\ c_t \geq 0, k_{t+1} \geq 0 & \forall t \geq 0 \end{cases}$$

- k_t is a **state variable** $\longrightarrow$ results from past decisions & law of motion

Need to add the transversality condition to define the problem properly...

The sequence problem

$$V_0(k_0) = \max_{\{c_t, k_{t+1}\}_{\forall t \geq 0}} \sum_{t \geq 0} \beta^t u(c_t) \quad \text{s.t.} \begin{cases} c_t + k_{t+1} = f(k_t) + (1-\delta)k_t & \forall t \geq 0 \\ k_0 \text{ given} & \\ c_t \geq 0, k_{t+1} \geq 0 & \forall t \geq 0 \end{cases}$$

- k_t is a **state variable** $\longrightarrow$ results from past decisions & law of motion
- (c_t, k_{t+1}) are **control variables** $\longrightarrow$ decisions at each t
  - ↪ Chosen at each period by the household, given the state and feasibility constraints
  - → They control for the evolution of the state variable

Need to add the transversality condition to define the problem properly...

The sequence problem

$$V_0(k_0) = \max_{\{c_t, k_{t+1}\}_{\forall t \geq 0}} \sum_{t \geq 0} \beta^t u(c_t) \quad \text{s.t.} \begin{cases} c_t + k_{t+1} = f(k_t) + (1-\delta)k_t & \forall t \geq 0 \\ k_0 \text{ given} \\ c_t \geq 0, k_{t+1} \geq 0 & \forall t \geq 0 \end{cases}$$

- k_t is a **state variable** $\longrightarrow$ results from past decisions & law of motion
- (c_t, k_{t+1}) are **control variables** $\longrightarrow$ decisions at each t
  - ↪ Chosen at each period by the household, given the state and feasibility constraints
  - → They control for the evolution of the state variable
- $V_0(k_0)$ is the value function of the household at time 0
  - ↪ Discounted sum of utility streams, given optimal **sequence of controls** and state k_0

Need to add the transversality condition to define the problem properly...

Solving the problem in the sequence space – shooting

How to solve this problem? $\Rightarrow$ let's mix analytical + computational methods . . .

Solving the problem in the sequence space – shooting

How to solve this problem? $\Rightarrow$ let's mix analytical + computational methods ...

1. This problem can be reduced to a **constrained** system of difference equations

$$\{\text{Euler Equation, Resource Constraint}\} \quad \text{s.t.} \quad \{c \geq 0, k \geq 0\} \quad \text{and} \quad k_0 \text{ given}$$

Solving the problem in the sequence space – shooting

How to solve this problem? $\Rightarrow$ let's mix analytical + computational methods ...

1. This problem can be reduced to a **constrained** system of difference equations

$$\{\text{Euler Equation, Resource Constraint}\} \quad \text{s.t.} \quad \{c \geq 0, k \geq 0\} \quad \text{and} \quad k_0 \text{ given}$$

$$\{c_{t+1} = g_1(k_t, c_t), k_{t+1} = g_2(k_t, c_t)\} \quad \text{s.t.} \quad \{c \geq 0, k \geq 0\} \quad \text{and} \quad k_0 \text{ given}$$

Solving the problem in the sequence space – shooting

How to solve this problem? $\Rightarrow$ let's mix analytical + computational methods ...

1. This problem can be reduced to a **constrained** system of difference equations

$$\{\text{Euler Equation, Resource Constraint}\} \quad \text{s.t.} \quad \{c \geq 0, k \geq 0\} \quad \text{and} \quad k_0 \text{ given}$$

$$\{c_{t+1} = g_1(k_t, c_t), k_{t+1} = g_2(k_t, c_t)\} \quad \text{s.t.} \quad \{c \geq 0, k \geq 0\} \quad \text{and} \quad k_0 \text{ given}$$

2. Get a sequence $\{c_{t+1}, k_{t+1}\}_{\forall t \leq T}$ that depends on $\{k_0, c_0\}$

Solving the problem in the sequence space – shooting

PARIS SCHOOL OF ECONOMICS
ECOLE D'ECONOMIE DE PARIS

How to solve this problem? $\Rightarrow$ let's mix analytical + computational methods ...

1. This problem can be reduced to a **constrained** system of difference equations

$$\{\text{Euler Equation, Resource Constraint}\} \quad \text{s.t.} \quad \{c \geq 0, k \geq 0\} \quad \text{and} \quad k_0 \text{ given}$$

$$\{c_{t+1} = g_1(k_t, c_t), k_{t+1} = g_2(k_t, c_t)\} \quad \text{s.t.} \quad \{c \geq 0, k \geq 0\} \quad \text{and} \quad k_0 \text{ given}$$

2. Get a sequence $\{c_{t+1}, k_{t+1}\}_{\forall t \leq T}$ that depends on $\{k_0, c_0\}$

3. **Shooting**: Find c_0 such that the system meets a **terminal condition**

Infinite time case (truncated to T)

Finite time case

Solving the problem in the sequence space – shooting

PARIS SCHOOL OF ECONOMICS
ECOLE D'ECONOMIE DE PARIS

How to solve this problem? $\Rightarrow$ let's mix analytical + computational methods ...

1. This problem can be reduced to a **constrained** system of difference equations

$$\{\text{Euler Equation, Resource Constraint}\} \quad \text{s.t.} \quad \{c \geq 0, k \geq 0\} \quad \text{and} \quad k_0 \text{ given}$$

$$\{c_{t+1} = g_1(k_t, c_t), k_{t+1} = g_2(k_t, c_t)\} \quad \text{s.t.} \quad \{c \geq 0, k \geq 0\} \quad \text{and} \quad k_0 \text{ given}$$

2. Get a sequence $\{c_{t+1}, k_{t+1}\}_{\forall t \leq T}$ that depends on $\{k_0, c_0\}$

3. **Shooting**: Find c_0 such that the system meets a **terminal condition**

Infinite time case (truncated to T)

$$k_{T+1} = k_T \quad \text{and} \quad c_{T+1} = c_T$$

Need T to be large enough! ($\lim_{t \to \infty} \beta^t = 0$)

Finite time case

Solving the problem in the sequence space – shooting

How to solve this problem? $\Rightarrow$ let's mix analytical + computational methods ...

1. This problem can be reduced to a **constrained** system of difference equations

$$\{\text{Euler Equation, Resource Constraint}\} \quad \text{s.t.} \quad \{c \geq 0, k \geq 0\} \quad \text{and} \quad k_0 \text{ given}$$

$$\{c_{t+1} = g_1(k_t, c_t), k_{t+1} = g_2(k_t, c_t)\} \quad \text{s.t.} \quad \{c \geq 0, k \geq 0\} \quad \text{and} \quad k_0 \text{ given}$$

2. Get a sequence $\{c_{t+1}, k_{t+1}\}_{\forall t \leq T}$ that depends on $\{k_0, c_0\}$

3. **Shooting**: Find c_0 such that the system meets a **terminal condition**

Infinite time case (truncated to T)

$$k_{T+1} = k_T \quad \text{and} \quad c_{T+1} = c_T$$

Need T to be large enough! ($\lim_{t\to\infty} \beta^t = 0$)

Finite time case

$$k_{T+1} = 0$$

Solving the problem in the sequence space – shooting

How to solve this problem? $\Rightarrow$ let's mix analytical + computational methods …

1. This problem can be reduced to a **constrained** system of difference equations

$$\{\text{Euler Equation, Resource Constraint}\} \quad \text{s.t.} \quad \{c \geq 0, k \geq 0\} \quad \text{and} \quad k_0 \text{ given}$$

$$\{c_{t+1} = g_1(k_t, c_t), k_{t+1} = g_2(k_t, c_t)\} \quad \text{s.t.} \quad \{c \geq 0, k \geq 0\} \quad \text{and} \quad k_0 \text{ given}$$

2. Get a sequence $\{c_{t+1}, k_{t+1}\}_{\forall t \leq T}$ that depends on $\{k_0, c_0\}$

3. **Shooting**: Find c_0 such that the system meets a **terminal condition**

Infinite time case (truncated to T)

$$k_{T+1} = k_T \quad \text{and} \quad c_{T+1} = c_T$$

Need T to be large enough! ($\lim_{t\to\infty} \beta^t = 0$)

Finite time case

$$k_{T+1} = 0$$

Challenge: how to find the right c_0? Objective of this week!

Deriving the system of first-difference equations

1. Set up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \Big[u(c_t) + \lambda_t \big(f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \big) \Big].$$

Deriving the system of first-difference equations

1. Set up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \Big[u(c_t) + \lambda_t \big(f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \big) \Big].$$

2. FOCs:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \big(u'(c_t) - \lambda_t \big) = 0 \qquad \Longrightarrow \qquad u'(c_t) = \lambda_t,$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \big(f'(k_{t+1}) + 1 - \delta \big) = 0 \qquad \Longrightarrow \qquad \lambda_t = \beta \lambda_{t+1} \big(f'(k_{t+1}) + 1 - \delta \big).$$

Deriving the system of first-difference equations

1. Set up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left[u(c_t) + \lambda_t \left(f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \right) \right].$$

2. FOCs:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \left(u'(c_t) - \lambda_t \right) = 0 \quad \implies \quad u'(c_t) = \lambda_t,$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \left(f'(k_{t+1}) + 1 - \delta \right) = 0 \quad \implies \quad \lambda_t = \beta \lambda_{t+1} \left(f'(k_{t+1}) + 1 - \delta \right).$$

3. Combine the two FOCs to get the Euler equation. It yields the following system of difference equations:

System of difference equations

$$\forall t \in \{1, T\} \quad \begin{cases} c_{t+1} = (u')^{-1} \left(\dfrac{u'(c_t)}{\beta \left(f'(k_{t+1}) + 1 - \delta \right)} \right) \\ k_{t+1} = f(k_t) + (1-\delta)k_t - c_t \end{cases} \quad \text{and} \quad \begin{cases} k_0 \text{ given} \\ \text{Terminal condition holds at } T \end{cases}$$

Coding the system in Matlab

From the first exercise, you should should know how to code a forward simulation of the system in Matlab…

$\Rightarrow$ Let's do it on the whiteboard!

Multiple shooting

PARIS SCHOOL OF ECONOMICS
ECOLE D'ECONOMIE DE PARIS

Use the structure of the problem to inform your choice of c_0

1. Unique steady state and saddle-path stability $\Rightarrow$ unique solution within boundaries

2. Guess c_0, project the system forward until the boundary condition is respected at time $T + 1$ ▸ intuition

Multiple shooting

Use the structure of the problem to inform your choice of c_0

1. Unique steady state and saddle-path stability $\Rightarrow$ unique solution within boundaries
2. Guess c_0, project the system forward until the boundary condition is respected at time $T+1$ ▸ intuition
   - If $k_{T+1} < k_{T+1}^{\text{boundary}}$, then savings were too low initially $\rightarrow$ decrease c_0
   - If $k_{T+1} > k_{T+1}^{\text{boundary}}$, then savings were too high initially $\rightarrow$ increase c_0

Multiple shooting

Use the structure of the problem to inform your choice of c_0

1. Unique steady state and saddle-path stability $\Rightarrow$ unique solution within boundaries

2. Guess c_0, project the system forward until the boundary condition is respected at time $T+1$ ▸ intuition

   - If $k_{T+1} < k_{T+1}^{\text{boundary}}$, then savings were too low initially $\rightarrow$ decrease c_0
   - If $k_{T+1} > k_{T+1}^{\text{boundary}}$, then savings were too high initially $\rightarrow$ increase c_0

$\Rightarrow$ Rootfinding on the boundary condition error (e.g. bisection)!

$$\mathtt{h}(c_0) = k_{T+1}(c_0) - k_{T+1}^{\text{boundary}} \qquad \text{Scaled error:} \quad \theta(c_0) = \mathtt{h}(c_0)/\bar{k}$$

Multiple shooting

Use the structure of the problem to inform your choice of c_0

1. Unique steady state and saddle-path stability $\Rightarrow$ unique solution within boundaries
2. Guess c_0, project the system forward until the boundary condition is respected at time $T+1$ ▸ intuition
   - If $k_{T+1} < k_{T+1}^{\text{boundary}}$, then savings were too low initially $\rightarrow$ decrease c_0
   - If $k_{T+1} > k_{T+1}^{\text{boundary}}$, then savings were too high initially $\rightarrow$ increase c_0

$\Rightarrow$ Rootfinding on the boundary condition error (e.g. bisection)!

$$\mathtt{h}(c_0) = k_{T+1}(c_0) - k_{T+1}^{\text{boundary}} \qquad \text{Scaled error:} \quad \theta(c_0) = \mathtt{h}(c_0)/\bar{k}$$

Be careful: Account for feasibility constraints, at each t: $c_t, k_t \geq 0$, update the $\mathtt{h}$ function accordingly!

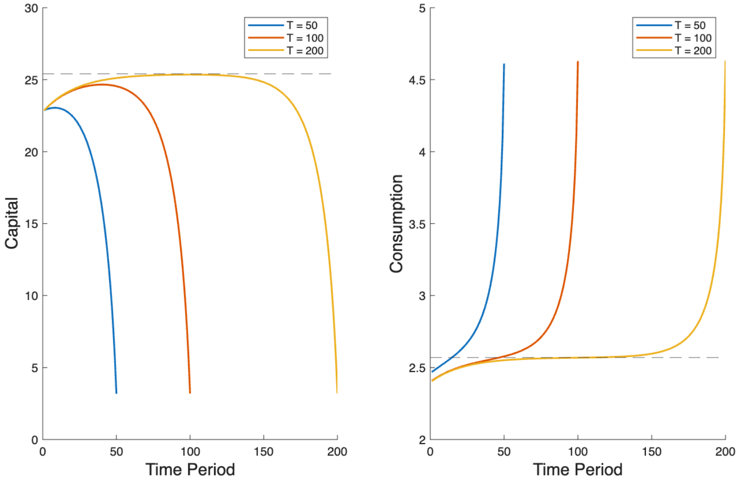
Any idea? $\rightarrow$ update directly the error with the right sign when a constraint is violated!

Example – Expected Results when T is finite

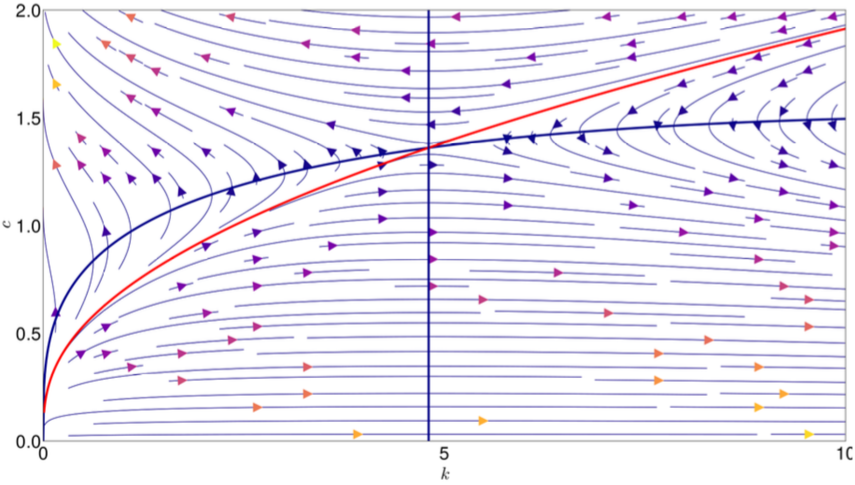
What should be the path of capital over time? If T is small? If T is large?

Example – Expected Results when T is finite

What should be the path of capital over time? If T is small? If T is large?



Phase diagram



Back