Quantitative Macroeconomics I
Tutorial Session 1

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Today's Agenda

1/ Framework: Neoclassical Growth Model

- Model presentation & main equations
- System of difference equations & phase diagram

2/ How to solve the model using a shooting algorithm?

$\Rightarrow$ First problem set to be solved for October 3, noon!

Link to submit your Problem Set: Google Classroom

The Ramsey Growth Model

- The Ramsey model is the basis of all modern macroeconomic models, and is simple!
  - Representative agent model with endogenous saving rate
  - Perfect competition & no friction: decentralized solution = social planner solution
  - Consider a discrete time version $t \in \mathbb{N}_+$
  - Parameters: $\alpha \in (0, 1)$, $\beta \in (0, 1)$, $\delta \in (0, 1)$, $\sigma \in \mathbb{R}_+ \backslash \{1\}$

- Perfectly competitive firm produces a generic good from capital $f(k) = k^\alpha$

- Representative household maximizes its flows of utility over time: $\sum_{t=0}^{\infty} \beta^t u(c_t)$

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

- Resource constraint: $c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$

The sequence problem

$$V_0 = \max_{\{c_t, k_{t+1}\}_{\forall t \geq 0}} \sum_{t \geq 0} \beta^t u(c_t) \quad \text{s.t.} \begin{cases} c_t + k_{t+1} = f(k_t) + (1-\delta)k_t & \forall t \geq 0 \\ k_0 \text{ given} \\ c_t \geq 0, k_{t+1} \geq 0 & \forall t \geq 0 \end{cases}$$

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  - ↪ Chosen at each period by the household, given the state and feasibility constraints
  - → They control for the evolution of the state variable
- V_0 is the value function of the household at time 0
  - ↪ Discounted sum of utility streams, given optimal **sequence of controls**

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3. **Shooting**: Find c_0 such that for T sufficiently large, the system reaches a steady state

$$k_{T+1} = k_T \quad \text{and} \quad c_{T+1} = c_T$$

$\Rightarrow$ Challenge: how to find the right c_0? Objective of your first problem set!

A system of difference equations

on the board explanation...

Get the sequence

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Shooting: Agnostic or informed?

Informed shooting: use the structure of the problem to inform your choice of c_0

1. Unique steady state and saddle-path stability $\Rightarrow$ only 1 solution within boundaries
2. Guess any c_0, project the system forward until boundary conditions are met ▸ intuition

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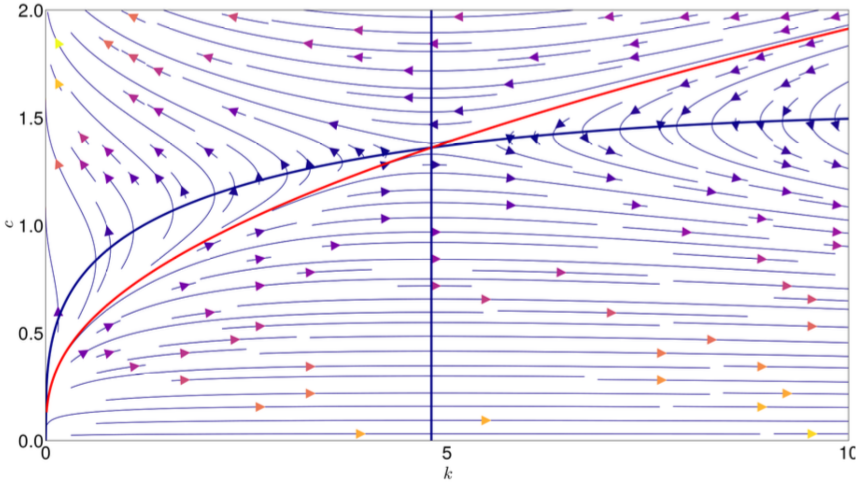
Phase diagram



Back